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The worst visibility walk in a random Delaunay triangulation is $O(\sqrt{n})$

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Abstract: We show that the memoryless routing algorithms Greedy Walk, Compass Walk, and all variants of visibility walk based on orientation predicates are asymptotically optimal in the average case on the Delaunay triangulation. More specifically, we consider the Delaunay triangulation of an unbounded Poisson point process of unit rate and demonstrate that the worst-case path between any two vertices inside a domain of area n has a number of steps that is not asymptotically more than the shortest path which exists between those two vertices with probability converging to one (as long as the vertices are sufficiently far apart.) As a corollary, it follows that the worst-case path has $O(\sqrt{n})$ steps in the limiting case, under the same conditions. Our results have applications in routing in mobile networks and also settle a long-standing conjecture in point location using walking algorithms. Our proofs use techniques from percolation theory and stochastic geometry.

Key-words: Probabilistic analysis – Worst-case analysis – Walking algorithms

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La pire marche par visibilité dans une triangulation de Delaunay de points aléatoires est en $O(\sqrt{n})$

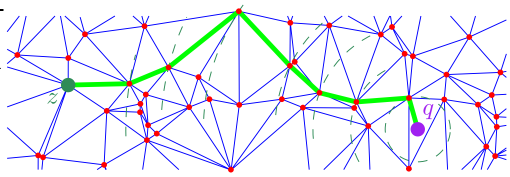
Résumé : Nous montrons que les algorithmes de routage sans mémoire de marche gloutonne, de marche au compas et toutes les variantes de marche par visibilité sont asymptotiquement optimale en moyenne pour la triangulation de Delaunay. Plus précisément, nous considérons la triangulation de Delaunay d'un processus de Poisson non borné d'intensité un et démontrons que le rapport entre les nombre d'étapes du pire et du meilleur chemin entre deux sommets suffisamment loin dans un domaine d'aire n est borné par une constante avec une probabilité convergeant vers 1. On en déduit comme corollaire que le pire chemin a au plus $O(\sqrt{n})$ étapes. Ce résultat a des applications au routage dans les réseaux mobiles et réponds à une conjecture sur les algorithmes de localisation par marche dans les triangulations. Nos démonstrations utilisent des résultats de percolation et de géométrie stochastique.

Mots-clés : Analyse probabiliste – Analyse dans le cas le pire – Algorithmes de marche

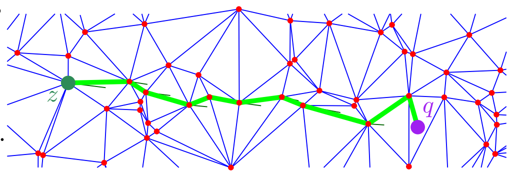
1 Introduction

Given a graph $G = G(V, E)$ embedded in d-dimensional Euclidean space $\mathbb{R}^d$, we define *graph navigation* to be the problem of finding a path from a source node, $z \in V$ to the nearest neighbour of a destination node, $q \in \mathbb{R}^d$ using only information gained from the vertices visited by the algorithm. This problem is highly related to the problems of geometric routing in networks and also point location in geometric data structures [9]. For brevity, we call graph navigation algorithms *walking algorithms*, and we say a walking algorithm is *memoryless* if every successive step can be computed without knowledge of the history of the walk. In this paper, we focus on the case of the Delaunay triangulation in $\mathbb{R}^2$. In the context of geometric routing in networks,[1] the Delaunay triangulation has been proposed as a network topology due to it acting as a spanner and because it can be constructed locally [2, 11, 14] (although not from unit disk graph [4]). In addition, computational libraries such as CGAL [7] which represent Delaunay triangulations as data structures make use of walking algorithms to efficiently perform point location.

Worst-case complexity bounds for walking algorithms tend to be pessimistic, since examples can easily be constructed in which algorithms visit every vertex in the graph at least once. Despite this, one might expect the algorithms to require 'approximately' $O(n^{1/d})$ steps to terminate in a triangulation of n points in $\mathbb{R}^d$ under some normality conditions. Such a result was conjectured in the very first papers using walking algorithms for point location [5, 12, 15], however it has only been proved formally in the *average-case* setting of n random points in a square for the algorithms Straight Walk [10], which visits every triangle intersecting the line segment between the initial point and destination, and Cone Walk [6], which successively chooses points directed towards the destination. Unfortunately, the algorithms which are most used in practice remain unanalysed, often because they are memoryless, and so exhibit strong dependence between steps. In this paper, we resolve this situation by proving that the conjecture is true for all of the most commonly used memoryless walking algorithms given the Delaunay triangulation of a Poisson point process observed in a window of area n. The bounds we achieve are in fact much stronger than the original conjecture, since we show that *no asymptotically sub-optimal path exists* for n large enough. We are confident that the methods we employ may be recycled to achieve similar bounds on a far greater variety of geometric structures and walking algorithms. The particular algorithms we focus on in this paper are outlined below.



Greedy Walk *Greedy Walk* is perhaps the simplest graph navigation algorithm. The algorithm is initialised by some vertex $z \in V$, and then iteratively chooses the vertex that is closest to the destination among the neighbours of the current vertex at each step. It is easy to see that Greedy Walk always succeeds on the Delaunay triangulation. Namely, assume that the walk ended at v and the disk D of centre q trough v is not empty (v is not the nearest neighbor); then one can construct an empty disk tangent to D at v passing through another vertex w closer to q, thus vw is a Delaunay edge giving a contradiction.

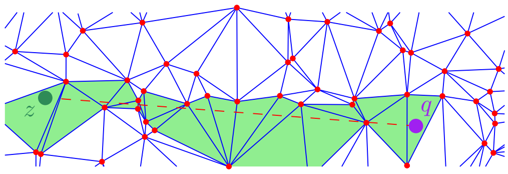


Compass Walk Given a current vertex u and a destination vertex q, *Compass Walk* is a deterministic memoryless graph navigation algorithm that works by choosing one of the two neighbours, v_1, v_2 of u where uv_1v_2 is the triangle intersected by ray uq. Often, the edge with the smallest angle with uq is preferred, but choosing randomly or based on the length of the line

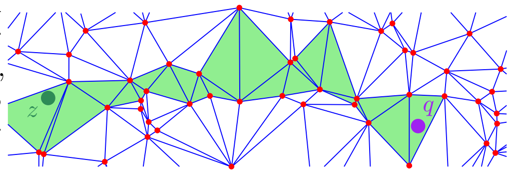
[1]For example Mobile Ad-hoc NETworks or 'MANETS'

segment uv_i can lead to different behaviour on more general graphs [3]. All versions succeed on the Delaunay triangulation.

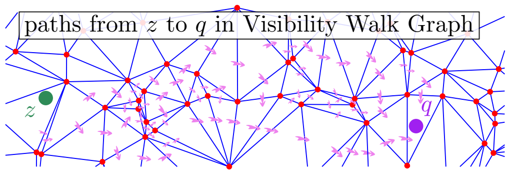
Straight Walk *Straight Walk* is a deterministic online graph navigation algorithm that succeeds in any triangulation. It works by visiting all triangles of the triangulation intersecting the line zq, for z the initial point and q the destination.



Visibility Walks *Visibility walk* is the best known walking algorithm that has historically been used for point location. It was first introduced by Lawson [15], and quickly adopted by others [5, 12] to speed-up the construction of Delaunay triangulations. As for Straight Walk the basic idea is to navigate a triangulation by moving between neighbouring triangles until the destination is reached. A visibility walk can go from a triangle t to its neighbour n if and only if the line supporting the common edge of n and t separates the interior of t from the query q. So the visibility walk is not unique since some triangles have two admissible successors. In such a case the successor can be chosen using any rule (e.g. at random, the first found, alternate counter clockwise and clockwise exploration…), in particular Straight Walk and all variants of Compass Walk are visibility walks.



Visibility Walk Graph We consider the dual graph of the triangulation, with dual edges oriented as described above for a destination q. A visibility walk corresponds to any oriented path in that graph. This graph has no cycles and a single sink: the triangle containing q.



Previous results

The straight walk was analysed in the average case and proved to be of expected complexity $\Theta(\sqrt{n})$ [1, 10]. However, it uses the fact that straight walk is not memoryless since z, the origin of the walk, must be known to decide the next triangle. In straight walk, the fact for a triangle to belong to the walk depends only on the segment zq and not on the rest of the walk making the dependence in probabilities easily tractable.

The case of memoryless walking algorithms is less easy. They may be treated as stochastic processes indexed by the number of steps visited, with random variables representing the *progress* of the walk at each step. The difficulty in most walking algorithms is that this sequence of random variables is neither independent and identically distributed, Markovian nor stationary, since a step in the walk may intersect a region in the process which has already been *seen* and because the distribution depends on the distance from the destination. This makes walking algorithms difficult to attack with traditional tools of probability theory. *Nearly-memoryless* walks were treated in this way by conditioning on special events to introduce independence by Broutin et al. [6]. This allowed a very accurate understanding of the walk process to be gained, with explicitly computed constants very close to what is actually observed in simulations. In addition, a tentative $O(\sqrt{n \log n})$ bound for the expected number of triangles visited by Visibility Walk was given by Zhu [20] using induction; although, unfortunately, the proof incorrectly assumes that each new point visited by the walk is independent of the history of the process. Applying this kind of step-by-step reasoning to walks such as Greedy Walk and visibility walks very quickly leads to ever-increasing numbers of cases that need to be considered separately. Certain pathological configurations of points make bounding these cases particularly difficult. Our technique is to attack the problem from another direction entirely. Instead of considering individual steps of walking algorithms, we 'zoom out' and focus instead on the properties of large *patches* of the point process, which may contain hundreds of individual triangles. This methodology will result in us sacrificing the fine control over the constants to obtain asymptotic order of magnitude. However, our proofs become significantly cleaner and most importantly, tractable.

2 Contributions

Given a locally finite set of points, $\mathbf{X}$ and a triangulation, $\mathrm{T}(\mathbf{X})$ (represented as a set of triples of points forming the triangles), we define the *visibility walk graph* with *destination* q, written $\mathbf{W}_q(\mathrm{T}(\mathbf{X}))$ to be the directed graph with *nodes* the triangles of $\mathrm{T}(\mathbf{X})$ and directed *arcs* representing the permissible steps that may be taken by visibility walks. Namely, given two triangles t and t' the arc is oriented from t to t' if q is on the same side as t' of the line supporting the common edge of t and t'. Let $\mathbf{P}^{(q)}(\mathrm{T}(\mathbf{X}))$ be the set of *paths* that may be constructed on the graph $\mathbf{W}_q(\mathrm{T}(\mathbf{X}))$, represented as sequences of triangles. It is well known (see also Corollary 10) that, for the Delaunay triangulation denoted $\mathrm{Del}(\mathbf{X})$, the visibility walk graph is *acyclic* so that every path in $\mathbf{W}_q(\mathrm{Del}(\mathbf{X}))$ converges on the node containing q. For $w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_n))$ we use $w(i)$ to denote the *ith* triangle in the walk w, $|w|$ its length ($q \in w(|w|)$), and $z(\sigma)$ the circumcenter of a triangle σ. We also similarly construct the set of paths that may be taken by Greedy Walk, which we denote $\mathbf{P}_{\mathrm{G}}^{(q)}(\mathrm{T}(\mathbf{X}))$. In this case, the nodes in the walk graph correspond to the vertices of $\mathrm{T}(\mathbf{X})$, and the set of paths forms a tree whose *root* is the nearest neighbour of q in $\mathbf{X}$.[2] This may be contrasted with visibility walks by noting that Greedy Walk is completely defined by the pair of initial and destination points, whereas visibility walks can make choices. Our main results are captured in the following theorems.

Theorem 1. *Let $\mathbf{X}$ be a homogeneous Poisson point process of rate 1 in $\mathbb{R}^2$ and a domain $D = [-\frac{\sqrt{n}}{2}, \frac{\sqrt{n}}{2}]$ of area n. Then for n sufficiently large,*

$$\mathbb{P}\left(\sup_{q\in\mathbf{D}} \sup_{\substack{w\in\mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_n)) \\ w(0)\in D}} \frac{|w|}{\|z(w(0)) - q\|\sqrt{n} + \log^3 n} \geq c_1 \right) \leq e^{-c_2\cdot\log^{\frac{3}{2}} n},$$

where c_1, c_2 are fixed positive constants.

Theorem 1 implies that the ratio between the shortest and longest possible paths between two given points for any kind of visibility walk is bounded by a constant with high probability, as long as the distance between the destination and start point is longer than $\log^3 n$, in the asymptotic limit.

Corollary 2. *Under the same conditions as Theorem 1,*

$$\lim_{n\to\infty} \sup_{q\in\mathbf{D}} \sup_{\substack{w\in\mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_n)) \\ w(0)\in D}} \frac{|w|}{\sqrt{n}} \leq c_1.$$

This corollary is a direct consequence of the Borel-Cantelli Lemma.

Corollary 3. *Theorem 1 also holds verbatim for all variants of Compass Walk.*

Corollary 4. *Theorem 1 also holds for Greedy Walk with different constants.*

Corollaries 3 and 4 will be proven in Section 9.

[2] If the nearest neighbour is not unique, the graph is a forest.

Sketch of proof

We first have to choose a way of measuring the progress in visibility walks. We choose to use the power of the query point with respect to the circle circumscribing the current triangle. This measure is strictly decreasing during the walk (see Section 6). This means that if we can find an event that gives 'good' progress in the circle power and which occurs sufficiently frequently, we can easily bound the number of steps required for a given walk. Applying this strategy *incrementally* as the walk progresses is the obvious way to proceed, however despite the authors' best efforts, untangling the dependence between the steps proved to be consistently unsuccessful.

The solution we present here is a result of 'rethinking' the formulation of the problem: instead of considering the walk as a sequence of steps, we consider all possible paths simultaneously. This is done by considering a regular grid whose cells contains on average a, reasonably big, constant number of points. We say that a grid cell is good if it does not intersect a Voronoi cell that span several grid cells and if all the triangles inside the cell induce a substantial progress in the circle power. These has two main advantages: the goodness of grid cells is independent if the cells are not neighbours in the grid and the number of possible paths in the grid is much smaller than in the triangulation.

In Section 4 we state our results for walks in a square of $k \times k$ grid cells with a Poisson point process of a well chose intensity, this result is proved in Sections 4 to 8.

In Section 5 we study the size of paths in a grid (also called *lattice animals*) using theory of percolation. We prove that if the badness probability is below some threshold, there is no big walk in the grid with high probability.

In Section 6 we relate the progress in the power of tof the query point with respect to the circle circumscribing the current triangle.he query point with respect to the circle circumscribing the current triangle to the geometric parameters of the triangles.

In Section 7 we choose the grid size to tune correctly the goodness probability of a grid cell.

In Section 8 we exploit this results to actually prove that the absence of big walk in the grid translate to an absence of big walk on the Delaunay triangulation.

Figure 1 summarise the organisation of the proof of Theorem 1.

3 Preliminaries

For a triangle, $\sigma \in \mathrm{Del}(\mathbf{X})$, we denote its circumcentre $z(\sigma)$ and its circumradius $r(\sigma)$.

Let $\mathbf{G}$ be the graph whose set of vertices is $\mathbb{Z}^2$ and with edges

$$x \leftrightarrow y \quad \Longleftrightarrow \quad \|y - x\| = 1,$$

for all $x, y \in \mathbb{Z}^2$. $\mathbf{G}$ is known as a *lattice* or *grid* in the plane. We additionally write

$$\mathbf{B}(x,\ell) := \left\{ y \in \mathbb{Z}^2 : \|y - x\|_\infty \leq \ell \right\} \quad \text{and} \quad \partial\mathbf{B}(x,\ell) := \mathbf{B}(x,\ell) \setminus \mathbf{B}(x,\ell-1)$$

to refer to balls of vertices in the lattice. We denote the ball in $\mathbb{R}^2$ under the Euclidean and L_∞ norms as

$$B(x,\ell) := \left\{ y \in \mathbb{R}^2 : \|x - y\| \leq \ell \right\}, \qquad B_\infty(x,\ell) := \left\{ y \in \mathbb{R}^2 : \|x - y\|_\infty \leq \ell \right\}.$$

We define:

$$\left\{ C(\{v\}) : v \in \mathbb{Z}^2 \right\}, \qquad \text{where} \qquad C(\mathbf{A}) := \bigcup_{v \in \mathbf{A}} B_\infty(v, \tfrac{1}{2}); \quad \mathbf{A} \subseteq \mathbb{Z}^2, \tag{1}$$

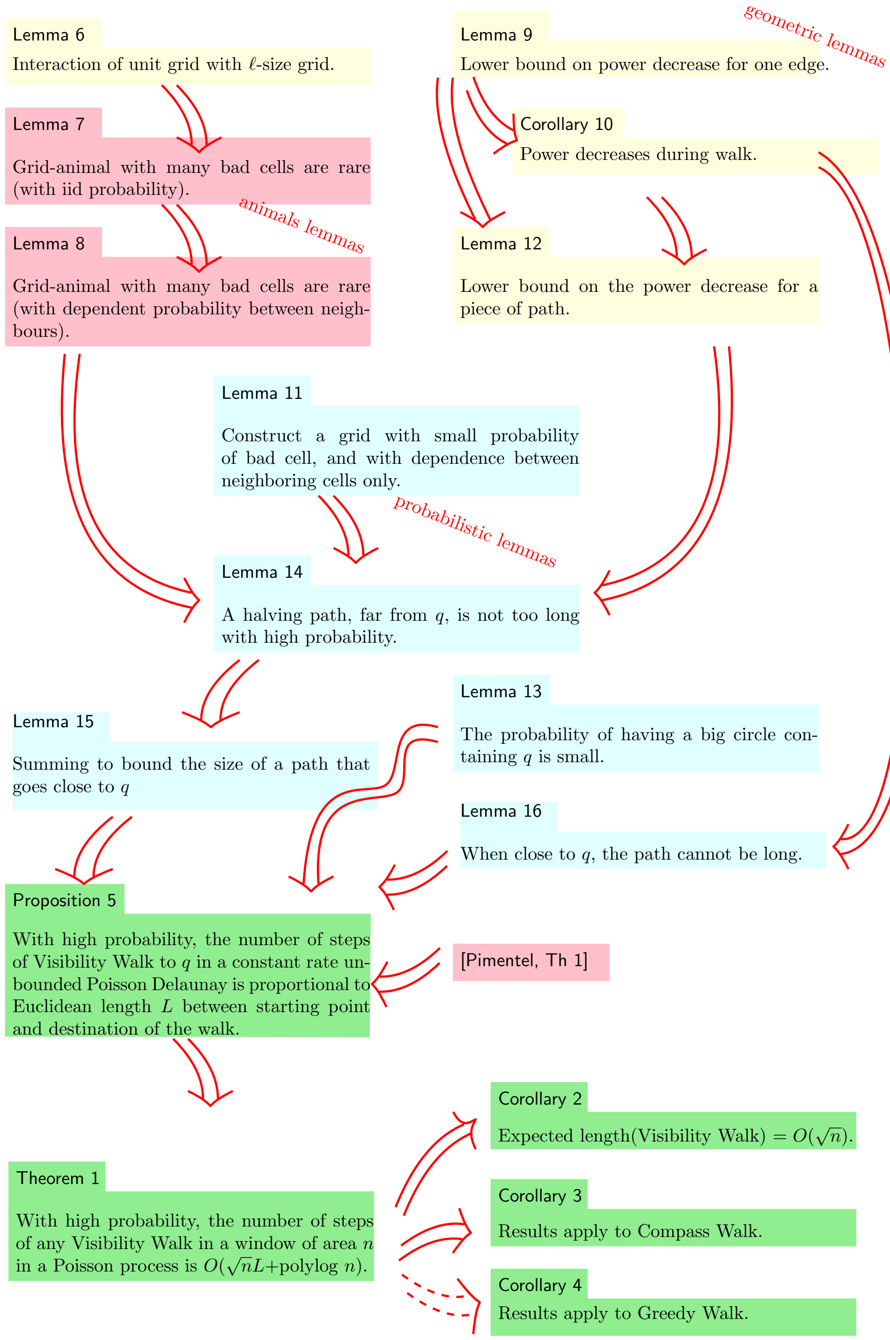


Figure 1: Proof organisation.

a collection of boxes covering $\mathbb{R}^2$ which is disjoint apart from the Lebesgue measure-zero borders of each box.[3] Many of our arguments will follow by associating independent and identically distributed random variables with each vertex in the lattice, $\mathbf{G}$. We denote these random variables as X_v for each $v \in \mathbb{Z}^2$. When $X_v \sim \text{Bernoulli}(p)$, this is also the standard model used in the theory of (site) percolation [13].

Lattice animals

A *lattice animal* is a collection of vertices $\mathbf{A} \subset \mathbb{Z}^2$ of the grid $\mathbf{G}$ such that for every pair of distinct vertices $u, v \in \mathbf{A}$ there is a path in $\mathbf{G}$ connecting u, v visiting only vertices in $\mathbf{A}$. Using digital geometry vocabulary, a lattice animal is a 4-connected subset of the grid, i.e. a connected set of pixels where a pixel can have 4 neighbours (horizontals or verticals). We denote by $|A|$ the size (or cardinality) of a lattice animal. The *size* of an animal is the number of vertices it contains. For $x \in \mathbb{Z}^2$, we shall write $\mathscr{A}_m^{(x)}$ to refer to the collection of all lattice animals of size m containing x. A lattice animal $\mathbf{A} \in \mathscr{A}_m^{(x)}$ is called *greedy* if it maximises the sum over the random variables X_v for $v \in \mathbf{A}$,

$$N(m) := \max_{\mathbf{A} \in \mathscr{A}_m^{(0)}} \sum_{v \in \mathbf{A}} X_v.$$

This maximum value has been studied in the literature [8] for various different distributions for the X_v, with extensions to the Poisson Voronoi tessellation by Pimentel and Rossignol [18]. Our interest in lattice animals is motivated by the following observation. We may *discretise* the path defined by some walk $w \in \mathbf{W}_q(\mathrm{T}(\mathbf{X}))$ by considering the collection of vertices $v \in \mathbb{Z}^2$ such that the box $C(v)$ intersects one of the line segments spanning the circumcentres of triangles in the walk. In other words, we consider all the grid cells traversed by the Voronoi path dual of w. Formally, we shall denote the sequence of Voronoi vertices visited by the walk as, $\bar{w} := \langle z(w(0)), z(w(1)), \ldots, z(w(|w|)) \rangle$. We abuse notation by identifying $\bar{w}$ with the piece-wise linear curve of line segments formed by $\bar{w}(i)\bar{w}(i+1)$ for $0 \leq i < |w|$. We then identify a lattice animal with the walk w as follows,

$$\mathbf{A}(\bar{w}) := \Big\{ v \in \mathbb{Z}^2 : C(v) \cap \bar{w} \neq \varnothing \Big\}.$$

4 Detailed statement of main result

Let $\mathbf{X}_\gamma$ be a homogenous Poisson process of rate γ in $\mathbb{R}^2$. As it is common in percolation theory, we will need to re-scale in order to fit the grid size with our needs. Instead of re-scaling the lattice, we shall re-scale the Poisson process using γ, since our notation will be simplified. On the process $\mathbf{X}_\gamma$, we construct the Delaunay triangulation, $\mathrm{Del}(\mathbf{X}_\gamma)$ which is almost surely in general position. In this section, we deal with the following proposition. We denote $V_{\mathbf{X}_\gamma}(x)$ the Voronoi cell of some point x. We also introduce the following notation to denote all walks starting from the 'ring' of boxes at distance k in the lattice from q,

$$\mathbf{P}_k^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) := \Big\{ w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) : \bar{w}(0) \in C(\partial \mathbf{B}(0, k)) \Big\}. \tag{2}$$

Proposition 5. *There exists γ such that for any $k \in \mathbb{N}$ large enough, $\mathbf{X}_\gamma$ the Poisson process on $\mathbb{R}^2$ with rate γ verifies:*

[3] We shall often abuse notation by writing $C(v)$ to mean $C(\{v\})$ for readability.

1. $\mathbb{P}\left(\exists w \in \mathbf{P}_k^{(0)}(\mathrm{Del}(\mathbf{X}_\gamma)) : |\mathbf{A}(\bar{w})| \geq C_1 k\right) \leq e^{-C_2\sqrt{k}}$,

2. $\mathbb{P}\left(\exists \mathbf{A} \in \mathscr{A}_m^{(0)}, m < C_1 k \,:\, \sum_{x\in\mathbf{X}_\gamma} \mathbb{1}_{V_{\mathbf{X}_\gamma}(x)\cap C(\mathbf{A})\neq\varnothing} \geq C_3 C_1 k\right) \leq e^{-C_4 k}$,

3. $\mathbb{P}\left(\exists w \in \mathbf{P}_k^{(0)}(\mathrm{Del}(\mathbf{X}_\gamma)) : |w| \geq C_3 C_1 k\right) \leq e^{-C_2\sqrt{k}} + e^{-C_4 k}$.

Where γ, C_1 *and* C_2, *are fixed positive constants which we explicitly compute in Lemma 11 and* C_3, C_4 *are fixed positive constants depending only on* γ.

Sections 5 to 8 will be devoted to the proof of Statement 1 in Proposition 5 which says that, with high probability, there is no visibility walk which induces a big lattice animal.

Proof of Proposition 5 Parts 2 and 3. Statement 2 says that, with high probability, the number of Voronoi cells intersecting an animal is proportional to the size of the animal. Statement 2 has been proved by Pimentel [17, Theorem 1]. Combining these two statements yields easily to Statement 3, since the Voronoi cells intersecting $C(\mathbf{A})$ is an upper bound for $|w|$. □

Proof of Theorem 1. A first remark is that the place of q in the $C(0)$ is not important, thus Proposition 5 to walks ending at any $q \in C(0)$.

Proposition 5 Part 3 can be rewritten in

$$\mathbb{P}\left((\exists q \in C(0) \exists w \in \mathbf{P}_k^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) :\ |w| \geq C_3 C_1 \log^3 n\right) \leq e^{-C_2 \log^{\frac{3}{2}} n}$$

using $k = \log^3 n$. When $k > \log^3 n$ we get

$$\begin{aligned}&\mathbb{P}\left((\exists q \in C(0) \exists w \in \mathbf{P}_k^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) :\ |w| \geq C_3 C_1 \|z(w(0)) - q\| \,|\, \|z(w(0)) - q\| \in [k-1, k]\right)\\ &\quad \leq e^{-C_2\sqrt{k}} + e^{-C_4 k} \leq e^{-C_2 \log^{\frac{3}{2}} n}\end{aligned}$$

Summing the last equation over $k \in [\log^3 n, \sqrt{n}]$ and adding the previous equation we get

$$\begin{aligned}&\mathbb{P}\left((\exists q \in C(0) \exists w \in \mathbf{P}_{\sqrt{n/2\gamma}}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) :\ |w| \geq C_3 C_1(\|z(w(0)) - q\| + \log^3 n)\right)\\ &\quad \leq \sqrt{n/2\gamma}\, e^{-C_2 \log^{\frac{3}{2}} n}\end{aligned}$$

Now, with $D_\gamma = \frac{1}{\gamma} D$,

$$\begin{aligned}&\mathbb{P}\left((\exists q \in D_\gamma \exists w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)), w(0) \in D_\gamma :\ |w| \geq C_3 C_1(\|z(w(0)) - q\| + \log^3 n)\right)\\ &\quad \leq \sum_w \mathbb{P}\left((\exists q \in C(w) \exists w \in \mathbf{P}_{2\sqrt{n/2\gamma}}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) :\ |w| \geq C_3 C_1(\|z(w(0)) - q\| + \log^3 n)\right)\\ &\quad \leq \frac{n}{\gamma} \cdot \sqrt{n/2\gamma}\, e^{-C_2 \log^{\frac{3}{2}} 4n}\end{aligned}$$

Since γ and C_i are constants, increasing the constant in the exponential make the polynomial term disappear and prove Theorem 1. □

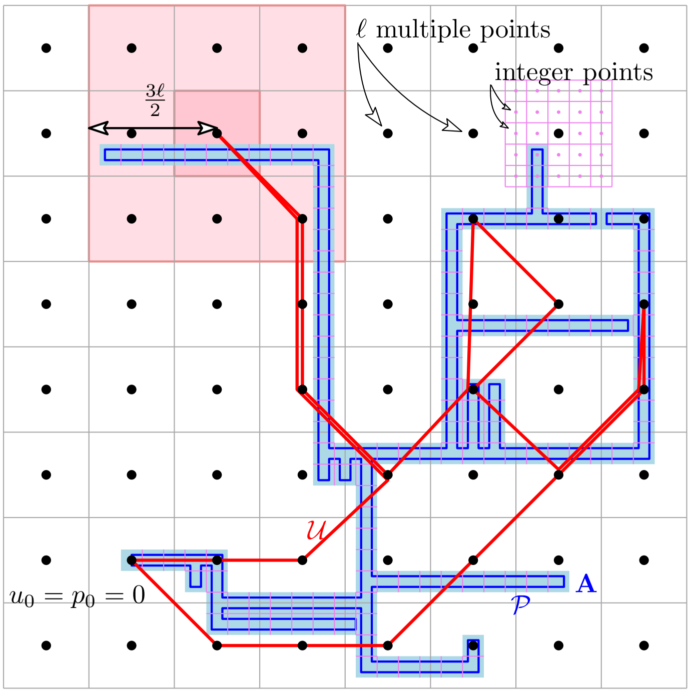


Figure 2: For the proof of Lemma 6

5 Size of Animals

Lemma 6. *For any animal* $\mathbf{A} \in \mathscr{A}_m^{(0)}$ *and* $1 \leq \ell \leq m$*, there exists a set of vertices* $u_0 = 0, u_1, \ldots u_r$ *forming a 8-connected* path *of* r *vertices in* $\mathbb{Z}^2$ *such that*

$$\mathbf{A} \subset \bigcup_{0 \leq i < r} \mathbf{B}\left(\ell u_i, \tfrac{3\ell}{2}\right) \qquad \text{and} \qquad r \leq \tfrac{2m-2}{\ell},$$

Proof. Similarly to Cox et al. [8, Lemma 1] we consider first a spanning tree on $\mathbf{A}$. A depth first traversal of this tree gives a path $\mathcal{P} = (p_j)$ in $\mathbf{G}$ of length $2m-2$ starting and finishing at the origin. $\mathcal{U} = (u_s)_{0 \leq s \leq r}$ is almost a discretisation of π in the grid of size ℓ (see Figure 2).

More precisely, we construct the path (u_s) incrementally. Assume that $(u_s)_{0 \leq s \leq t}$ discretizes $(p_j)_{0 \leq j \leq i}$, that is $\forall j < i$; $p_j \in \bigcup_{s=0}^{t} \mathbf{B}(\ell u_s, \frac{3\ell}{2})$, and $p_i \in \mathbf{B}(\ell u_t, \frac{\ell}{2})$. Now consider the point p_k with smallest index $k > i$ such that $p_k \notin \mathbf{B}(\ell u_t, \frac{3\ell}{2})$, then we define $u_{t+1} = \lfloor \frac{p_{k-1}}{\ell} \rceil$ the center of the square of side ℓ containing p_{k-1}. Notice that, since $p_{k-1} \in \mathbf{B}(\ell u_t, \frac{3\ell}{2})$, u_{i+1} is 8-connected to u_i.

We can lower bound $\|p_i p_{k-1}\|_\infty$:

$$\begin{aligned}
\|p_i p_k\|_\infty &> \ell \qquad \text{since } p_i \in \mathbf{B}(\ell u_t, \tfrac{\ell}{2}) \text{ and } p_k \notin \mathbf{B}(\ell u_t, \tfrac{3\ell}{2}) \\
\|p_i p_k\|_\infty &\geq \ell + 1 \\
\|p_i p_{k-1}\|_\infty &\geq \|p_i p_k\|_\infty - \|p_k p_{k-1}\|_\infty \geq \ell + 1 - 1 = \ell
\end{aligned}$$

and conclude that one step on $\mathcal{U}$ corresponds to at least ℓ steps in $\mathcal{P}$ and thus $|\mathcal{U}| = r \leq \left\lfloor \frac{|\mathcal{P}|}{\ell} \right\rfloor \leq \frac{2m-2}{\ell}$ (we can take the floor of $\frac{|\mathcal{P}|}{\ell}$ because the end of path $\mathcal{P}$ is covered by $\mathbf{B}(u_0, \frac{3\ell}{2})$ since

$p_{2m-2} = p_0 = 0$). □

The following lemma has been adapted from Lee [16]. Our proof closely follows the original, though in our case we seek strong bounds for a fixed m.

Lemma 7. *Associate with every $v \in \mathbf{G}$ a random variable X_v which is* Bernoulli(p) *distributed. Then for m sufficiently large and $p \le p_0 := \frac{1}{1500}$*

$$\mathbb{P}\left(\exists \mathbf{A} \in \mathscr{A}_m^{(0)} : \sum_{v \in \mathbf{A}} X_v \ge m\sqrt{\tfrac{p}{p_0}}\right) \le e^{-m\sqrt{p}}.$$

Proof. For $\mathbf{A} \in \mathscr{A}_m^{(0)}$, t and using Lemma 6 there exists a set of vertices $u_0, \dots u_r$ forming a 8-connected *path* of $r \le \frac{2m-2}{\ell}$ vertices in $\mathbb{Z}^2$ such that

$$\mathbf{A} \subset \bigcup_{i=0}^{r} \mathbf{B}(\ell u_i, \tfrac{3\ell}{2}),$$

with u_0=0. In the following, we choose $\ell := \left\lceil p^{-\frac{1}{2}} \right\rceil$. Doing this means we can bound the number of configurations, in particular, we have at most 8 ways of choosing the successor of u_i giving $8^r \le 8^{2m\sqrt{p}}$ ways to choose the $u_0, \dots, u_r$. Furthermore, it follows that, using $3\ell + 1 \le 3(p^{-\frac{1}{2}} + 1) + 1 = 3p^{-\frac{1}{2}}(1 + \frac{4}{3}\sqrt{p}) \le 3p^{-\frac{1}{2}}(1 + \frac{4}{3}\sqrt{p_0})$:

$$\left|\bigcup_{i=0}^{r} \mathbf{B}(\ell u_i, \tfrac{3\ell}{2})\right| \le r(3\ell+1)^2 \le 2m\sqrt{p} \cdot 9p^{-1}(1 + \tfrac{4}{3}\sqrt{p_0})^2 \le 18 \cdot mp^{-\frac{1}{2}}(1 + \tfrac{4}{3}\sqrt{p_0})^2.$$

Using the above bounds gives us,

$$\mathbb{P}\left(\exists \mathbf{A} \in \mathscr{A}_m^{(0)} : \sum_{v \in \mathbf{A}} X_v \ge m\sqrt{\tfrac{p}{p_0}}\right)$$

$$\le \sum_{u_0,\dots,u_r} \mathbb{P}\left(\sum_{v \in \bigcup_{i=0}^{r} \mathbf{B}(\ell u_i, \frac{3\ell}{2})} X_v \ge m\sqrt{\tfrac{p}{p_0}}\right) \tag{3}$$

$$\le \sum_{u_0,\dots,u_r} \exp(-m\sqrt{\tfrac{p}{p_0}}) \cdot \mathbb{E}\left[\prod_{v \in \bigcup_{i=0}^{r} \mathbf{B}(\ell u_i, \frac{3\ell}{2})} \exp(X_v)\right] \tag{4}$$

$$\le 8^{2m\sqrt{p}} \cdot \exp(-p_0^{-\frac{1}{2}} m\sqrt{p}) \cdot (1 - p + pe)^{18 \cdot mp^{-\frac{1}{2}}(1+\frac{4}{3}\sqrt{p_0})^2} \tag{5}$$

$$\le \exp(2\log 8 \cdot m\sqrt{p}) \cdot \exp(-p_0^{-\frac{1}{2}} m\sqrt{p}) \cdot \exp(p(e-1))^{18 \cdot mp^{-\frac{1}{2}}(1+\frac{4}{3}\sqrt{p_0})^2} \tag{6}$$

$$\le \exp\left(m\sqrt{p}(2\log 8 - p_0^{-\frac{1}{2}} + (e-1)18(1 + \tfrac{4}{3}\sqrt{p_0})^2)\right)$$

$$\le \exp(-m\sqrt{p}).$$

Where line (3) follows from the union bound, (4) follows since $\mathbb{P}(Z \ge t) \le e^{-t}\,\mathbb{E}[e^Z]$, for Z a non-negative random variable by the Markov inequality, (5) follows since the X_v are independent Bernoulli(p) distributed random variables, and (6) uses the fact that $\forall t, 1 + t \le e^t$. The value of p_0 has been chosen to ensure the last step. □

Lemma 7 is the driving force behind our proof. However, we cannot apply it directly, since regions in the Delaunay triangulation are always dependent on their neighbours. To deal with this, we will allow a *band* of local dependence around each region and then apply Lemma 7 multiple times with the total sum bounded by the union bound.

Lemma 8. *Consider the lattice* $\mathbf{G}$ *with associated random variables* X_v *for each* $v \in \mathbf{G}$ *such that* X_v *and* X_u *are* dependent *only if* $\|u - v\|_\infty = 1$*, and where the marginal distribution of* X_v *is* Bernoulli(p)*. Then if* $p \leq p_1 := \frac{1}{8000} = \frac{3}{16}p_0$*,*

$$\mathbb{P}\left(\exists \mathbf{A} \in \mathscr{A}_m^{(0)},\ \sum_{v\in\mathbf{A}} X_v \geq \tfrac{3m}{4}\right) \leq e^{-m\sqrt{p}/2},$$

for n *large enough.*

Proof. We begin by partitionning $\mathbf{G}$ in a collection of lattices, $\mathbf{G}_i$ for $i = 1, \ldots, 4$ whose vertex sets respectively are

$$I_i := 2\mathbb{Z}^2 + v_i, \quad \text{for} \quad v_i \in \{(0,0), (0,1), (1,0), (1,1)\}$$

with edges in lattice i given by,

$$x \leftrightarrow y \quad \Longleftrightarrow \quad \|x - y\| = 2,$$

for $x, y \in I_i$. Thus the vertices of each grid $\mathbf{G}_i$ are disjoint and the random variables X_v, X_u are all pairwise independent for any distinct $u, v \in I_i$. For a given $\mathbf{A} \in \mathscr{A}_m^{(0)}$, we may now define a collection of *sub-animals* by looking at the vertices of $\mathbf{G}_i$ that are in $\mathbf{A}$ or have one of their three "positive" neighbors that are in $\mathbf{A}$ (see Figure 3):

$$\mathbf{A}_i := \Big\{\, x \in I_i : \exists v_j \in \{(0,0), (0,1), (1,0), (1,1)\}\ x + v_j \in \mathbf{A} \,\Big\}.$$

This definition ensure the connectivity of the sub-animals. Since $\mathbf{A} \subset \cup \mathbf{A}_i$, it follows that

$$\sum_{v\in\mathbf{A}} X_v \leq \sum_{i\leq 4} \sum_{v\in\mathbf{A}_i} X_i,$$

and that $0 \leq |\mathbf{A}_i| \leq m$ for $i = 1, \ldots, 4$. We now note that for a collection of non-negative random variables $Z_1, \ldots, Z_k$, if $\sum_{i=1}^k Z_i \geq t$ then, there exists an i such that $Z_i \geq \frac{t}{k}$. So by the union bound,

$$\mathbb{P}\left(\exists \mathbf{A} \in \mathscr{A}_m^{(0)},\ \sum_{v\in\mathbf{A}} X_v \geq \tfrac{3m}{4}\right) \leq \sum_{i\leq 4} \mathbb{P}\left(\exists \mathbf{A} \in \mathscr{A}_m^{(0)},\ \sum_{v\in\mathbf{A}_i} X_v \geq \tfrac{3m}{16}\right). \tag{7}$$

Note that $|\mathbf{A}_i| \leq m$ and that the sum in the right hand side of (7) is strictly increasing in m. In particular,

$$\begin{aligned} &\Big\{\, \exists \mathbf{A} \in \mathscr{A}_m^{(0)},\ \exists i \in [1,4] : \sum_{v\in\mathbf{A}_i} X_v \geq \tfrac{3m}{16} \,\Big\} \\ &\qquad \Longrightarrow \quad \Big\{\, \exists i \in [1,4]\ \exists \mathbf{A} \in 2 \cdot \mathscr{A}_m^{(v_i)} : \sum_{v\in\mathbf{A}} X_v \geq \tfrac{3m}{16} \,\Big\}, \end{aligned} \tag{8}$$

where $k \cdot \mathscr{A}_m^{(x)}$ is taken to be the lattice $\mathscr{A}_m^{(x)}$ scaled by k. Since the scaling does not in anyway affect the bounds in Lemma 7, and since the X_v for all $v \in \mathbf{A}_i$ are all independent and identically Bernoulli(p) distributed by assumption, applying (7) and (8) and the fact that $p < p_1 = \frac{3}{16}p_0$ give a bound of $4e^{-m\sqrt{p}}$ that is smaller than $e^{-m\frac{\sqrt{p}}{2}}$ for m large enough. $\square$

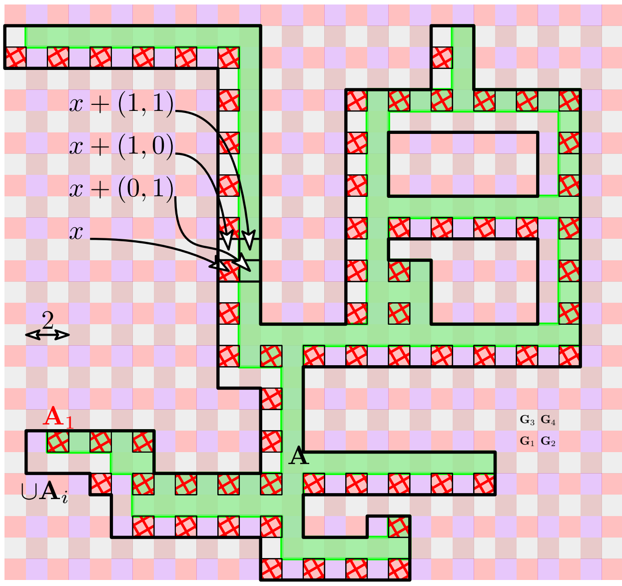


Figure 3: For the proof of Lemma 8. $\mathbf{A}$ is green, $\mathbf{A}_1$ is hatched red and $\mathbf{A}_1 \cup \mathbf{A}_2 \cup \mathbf{A}_3 \cup \mathbf{A}_4$ is shown by a black boundary.

6 Measuring Walk Progress

To measure the progress made by the walk, we have multiple options. An obvious choice is to use the Euclidean distance to q from one of the vertices or from the circumcentre of each triangle visited to the destination. However, it is not difficult to prove that none of these distances is decreasing during a walk, leading to extra conditions to be taken care of in our proofs. We shall thus focus instead on using the *circle power*. The circle power relative to a point q of a circle centred at $z \in \mathbb{R}^2$ and of radius r is defined to be $\|z - q\|^2 - r^2$. It is well known that, if a line through q cut the circle in z' and z'', the power is also $\overline{z'q} \cdot \overline{z''q}$ the product of the signed distance measured along the line, which is independent of the line. For a triangle σ and a point q, we shall say that σ has circle power

$$P(\sigma, q) := \|z(\sigma) - q\|^2 - r(\sigma)^2. \tag{9}$$

We remark that the power becomes negative when q is contained in $B(z(\sigma), r(\sigma))$. Helpfully, it turns out that when walking from a triangle $\sigma \in \mathrm{Del}(\mathbf{X})$ to its ith neighbour in a triangulation, $\mathrm{Del}(\mathbf{X})$, the change in circle power may be very simply expressed. We first define the following properties of a triangle $\sigma \in \mathrm{Del}(\mathbf{X}_\gamma)$,

$$\begin{aligned} d(\sigma) &:= \inf\Big\{ |z(\sigma) - z(\tau)| : \tau \in \mathrm{Del}(\mathbf{X}_\gamma) \setminus \sigma \Big\}, \\ \alpha_i(\sigma, q) &:= \angle\big(q, \sigma(i+1), \sigma(i+2)\big), \end{aligned} \tag{10}$$

where $\sigma(i)$ is taken to be the $(i \mod 3)th$ vertex of σ, which is opposite the ith triangle neighbouring σ (we assume that the vertices are ordered counterclockwise.)

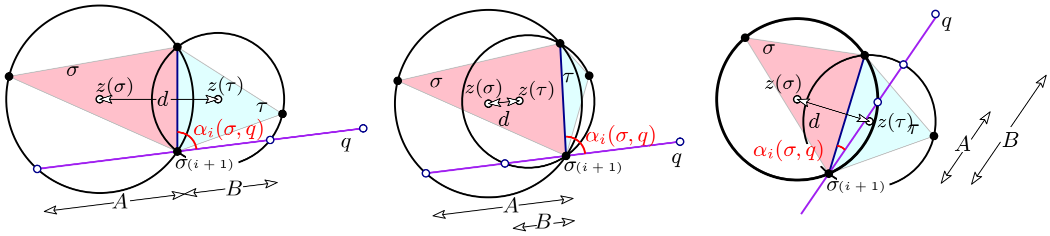


Figure 4: The change in circle power relative to q when moving between two overlapping circles is $-2d\sin\alpha\|pq\|$, and is thus independent of the radii.

Lemma 9. *The circle power, $\Delta_q(\sigma, i)$ relative to q when moving from a triangle σ to its ith neighbour, τ decreases by exactly,*

$$\Delta_q(\sigma, i) = P(\sigma, q) - P(\tau, q) = 2\sin\alpha_i(\sigma, q) \cdot \|z(\sigma)z(\tau)\| \cdot \|\sigma(i+1)q\|.$$

Proof. The proof rely on basic trigonometry. Referencing Figure 4, let A be the length of the part of line $q\sigma(i+1)$ that overlaps the interior of the circle circumscribing σ, and B be the length of the part of the line overlapping the interior of the circle circumscribing τ. If A and B do not overlap, the change in circle power when moving across the edge is now, denoting $\sigma(i+1) = p$ for short, given by

$$P(\sigma, q) - P(\tau, q) = \|pq\|(\|pq\| + A) - \|pq\|(\|pq\| - B) = \|pq\|(A + B).$$

To calculate $A+B$, we refer to Figure 4-left. Basic trigonometry gives us that $A + B = 2d\sin\alpha$ with $d = \|z(\sigma)z(\tau)\|$. If A and B do overlap on the side of p opposite to q, then

$$P(\sigma, q) - P(\tau, q) = \|pq\|(\|pq\| + A) - \|pq\|(\|pq\| + B) = \|pq\|(A - B).$$

again, basic trigonometry gives $A - B = 2d\sin\alpha$ (Figure 4-center). If A and B do overlap on the other side, then

$$P(\sigma, q) - P(\tau, q) = \|pq\|(\|pq\| - A) - \|pq\|(\|pq\| - B) = \|pq\|(B - A).$$

again, basic trigonometry gives $B - A = 2d\sin\alpha$ (Figure 4-right).

Notice that the position of q on the half line $\sigma(i+1)q$ does not play any role in the proof and the result hold even if q is inside one of the circle circumscribing σ or τ. □

Corollary 10. *The circle power is decreasing for visibility walks.*

Proof. It follows directly from Lemma 9 that $\Delta_q(\sigma, i) \geq 0$ when walking between two triangles according to the visibility walk definition. □

7 Getting Independence

By Lemma 9, we know that the diminution of circle power when leaving the triangle $\sigma \in \mathrm{Del}(\mathbf{X}_\gamma)$ is lower bounded by

$$|\Delta_q(\sigma, \cdot)| \geq 2(\|z(\sigma)q\| - r(\sigma)) \cdot d(\sigma) \cdot \min_{0<i\leq 3} \left\{ \sin\alpha_i(\sigma, q) \right\}, \tag{11}$$

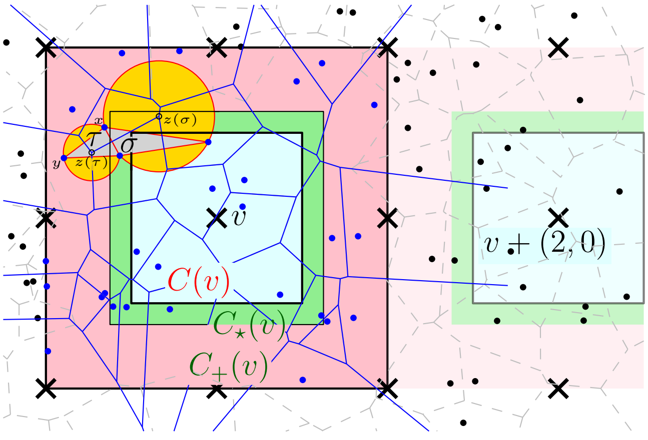


Figure 5: In blue the diagram of the points in $C_+(v)$, in dashed line the Voronoi diagram of all points.

as long as $\|z(\sigma)q\| \geq r(\sigma)$. We may ignore the first term now, since it will be explicitly bounded by the current position in the walk later. We instead focus on the *local* properties of σ. We define nested boxes around C_v: $C_\star(v) := B_\infty\left(v, \frac{5}{8}\right)$ and $C_+(v) := B_\infty\left(v, 0.99\right)$, and the following *bad* event indexed by the element of $\mathbf{G}$,

$$E_v := \bigcup_{\substack{\sigma\in\mathrm{Del}(\mathbf{X}_\gamma),\\ z(\sigma)\in C_\star(v)}} \left\{ \left\{ r(\sigma) \geq C_r \right\} \cup \left\{ d(\sigma) \leq C_d \right\} \cup \left\{ \min_{0<i\leq 3} \sin\alpha_i(\sigma, q) \leq C_\alpha \right\} \right\}, \tag{12}$$

The values C_r, C_d, C_α will be fixed at the end of the proof of the following lemma.

Lemma 11. *Let $p_2 \in (0, p_1]$ be fixed, then we can choose the constants $C_r \leq \frac{1}{8}$, $C_d \leq \frac{1}{2}C_r$, C_α in (12) and the intensity γ of the process $\mathbf{X}_\gamma$ such that the event E_v satisfies the following conditions,*

1. *E_v and E_w are independent if $\|v - w\| \geq 2$.*
2. *$\mathbb{P}(E_v) \leq p_2$*
3. *If event $(E_v)^c$ occurs, then for every $\sigma \in \mathrm{Del}(\mathbf{X}_\gamma)$ with $z(\sigma) \in C_\star(v)$ and $\|z(\sigma)q\| \geq C_r$,*

$$\Delta_q(\sigma, \cdot) \geq 2C_d \cdot C_\alpha \cdot (\|z(\sigma)q\| - C_r).$$

Proof of Lemma 11, Part (1) . Consider σ a Delaunay triangle of $\mathrm{Del}(\mathbf{X}_\gamma)$ with circumcenter $z(\sigma)$ inside $C_\star(v)$ and τ one of its Delaunay neighbor.

If $(E_v)^c$ occurs, and using $C_r \leq \frac{1}{8}$, $C_d \leq \frac{1}{16}$, we have that a vertex x of σ verifies

$$\|vx\|_\infty \leq \|vz(\sigma)\|_\infty + \sqrt{2}\|z(\sigma)x\| \leq \tfrac{5}{8} + \sqrt{2}C_r \leq \tfrac{5}{8} + \tfrac{\sqrt{2}}{8} \simeq 0.802 \leq 0.99$$

and a vertex y of τ, with x common to σ and τ, verifies

$$\begin{aligned} \|z(\tau)y\| =& \|z(\tau)x\| \leq \|z(\tau)z(\sigma)\| + \|z(\sigma)x\|, \\ \|vy\|_\infty \leq& \|vz(\sigma)\|_\infty + \sqrt{2}\left(\|z(\sigma)z(\tau)\| + \|z(\tau)y\|\right) \\ \leq& \tfrac{5}{8} + \sqrt{2}(C_d + C_d + C_r) \leq \tfrac{5}{8} + \sqrt{2}\tfrac{1+1+2}{16} \simeq 0.979 \leq 0.99 \end{aligned}$$

and the vertices of σ and τ are inside $C_+(v)$ (see Figure 5). Thus $z(\sigma)$ and $z(\tau)$ are also Voronoi vertices of the dual of $\mathrm{Del}(\mathbf{X}_\gamma \cap C_+(v))$.

So, we can decide whether or not E_v occurs given only the information in the sigma-algebra of the probability space for $\mathbf{X}_\gamma$ restricted to events occurring in $C_+(v)$. This means that E_v, E_w are independent if $\|v - w\| \geq 2$ since $C_+(v) \cap C_+(w) = \varnothing$. □

Proof of Lemma 11, Part (2) . We begin by using the first-moment method to write the probability of the event E_v as the expectation of a sum,

$$\begin{aligned}\mathbb{P}(E_v) &= \mathbb{P}\left(\sum_{\substack{\sigma\in\mathrm{Del}(\mathbf{X}_\gamma),\\ z(\sigma)\in C_\star(v)}} \mathbb{1}_{r(\sigma)\geq C_r} + \sum_{\substack{\sigma\in\mathrm{Del}(\mathbf{X}_\gamma),\\ z(\sigma)\in C_\star(v)}} \mathbb{1}_{r(\sigma)<C_r}\, \mathbb{1}_{\cup_{i\leq 3}\sin\alpha_i(\sigma)\leq C_\alpha \cup d(\sigma)\leq C_d} \geq 1 \right)\\ &\leq E_v^{(r)} + E_v^{(d)} + E_v^{(\alpha)},\end{aligned}$$

where we define

$$\begin{aligned}E_v^{(r)} &:= \mathbb{E}\left[\sum_{\sigma\in\mathrm{Del}(\mathbf{X}_\gamma)} \mathbb{1}_{z(\sigma)\in C_\star(v)}\, \mathbb{1}_{r(\sigma)\geq C_r}\right],\\ E_v^{(d)} &:= \mathbb{E}\left[\sum_{\sigma\in\mathrm{Del}(\mathbf{X}_\gamma)} \mathbb{1}_{z(\sigma)\in C_\star(v)}\, \mathbb{1}_{r(\sigma)<C_r}\, \mathbb{1}_{d(\sigma)\leq C_d}\right],\\ E_v^{(\alpha)} &:= \mathbb{E}\left[\sum_{\sigma\in\mathrm{Del}(\mathbf{X}_\gamma)} \mathbb{1}_{z(\sigma)\in C_\star(v)}\, \mathbb{1}_{r(\sigma)<C_r}\, \mathbb{1}_{\cup_{i\leq 3}\sin\alpha_i(\sigma)\leq C_\alpha}\right].\end{aligned}$$

It suffices to bound the given expectations. In each case our method will follow the same 'recipe' of applying the Slivnyak-Mecke formula (see, for example, Schneider and Weil [19, Corollary 3.2.3]) and a Blaschke-Petkanschin change of variables, which maps three points of a triangle to the centre and radius of its circumcircle plus three angles (see, for example, Schneider and Weil [19, Theorem 7.3.1]). Denoting x_1, x_2 ,x_3 the vertices of σ, we have,

$$\begin{aligned}E_v^{(r)} &= \tfrac{1}{3!}\,\mathbb{E}\left[\sum_{(x_1,x_2,x_3)\in(\mathbf{X}_\gamma)^3_{\neq}} \mathbb{1}_{B(\sigma)\cap\mathbf{X}_\gamma\setminus\{x_1,x_2,x_3\}=\varnothing}\, \mathbb{1}_{z(\sigma)\in C_\star(v)}\, \mathbb{1}_{r(\sigma)\geq C_r}\right]\\ &= \tfrac{1}{3!}\int_{(\mathbb{R}^2)^3} \mathbb{P}\Big(B(x_1,x_2,x_3)\cap\mathbf{X}_\gamma = \varnothing\Big)\mathbb{1}_{z(\sigma)\in C_\star(v)}\, \mathbb{1}_{r(\sigma)\geq C_r}\, \gamma^3 dx_1\, dx_2\, dx_3 \qquad (13)\\ &= \frac{\gamma^3}{3!}\int_{C_\star(v)}\int_{C_r}^{\infty}\int_{\mathbf{S}^3} e^{-\pi\gamma r^2}\cdot r^3 2\mathcal{A}(u_1,u_2,u_3)\,\mu(du_1)\,\mu(du_2)\,\mu(du_3)\,dr\,dz \qquad (14)\\ &= \frac{\gamma^3}{6}\cdot\left(\tfrac{10}{8}\right)^2 24\pi^2\int_{C_r}^{\infty} r^3\cdot e^{-\pi\gamma r^2}\,dr\\ &= \tfrac{25}{4}\pi^2\gamma^3\cdot\frac{1+\pi\gamma C_r^2}{\pi^2\gamma^2}e^{-\pi\gamma C_r^2} = \tfrac{25}{4}\gamma(1+\pi\gamma C_r^2)e^{-\pi\gamma C_r^2}\end{aligned}$$

Where $\mathcal{A}(u,v,w)$ is the area of the convex hull of $\{u,v,w\}$ and $\mathbf{S}$ is the unit 1-sphere with associated uniform (Lebesgue) measure $\mu(\cdot)$. Equation (13) follows from the Slivnyak-Mecke formula, and Equation (14) uses the Blaschke-Petkanschin change of variables.

To deal with $E_v^{(d)}$, we first remark that a Voronoi vertex $z(\sigma)$ can have a short Voronoi edge $z(\sigma)z(\tau)$ of length $\leq C_d$ only if the ball $B\big(z(\sigma), r(\sigma)+2C_d\big)$ contains other points than the vertices of σ. To prove this, let's assume, w.l.o.g., that τ has vertices x_1, x_2, and y and assuming $\|z(\sigma)z(\tau)\| \leq C_d$, we will prove that $y \in B\big(z(\sigma), r(\sigma)+2C_d\big) \neq \varnothing$. We have:

$$\begin{aligned}\|z(\sigma)y\| \leq& \|z(\sigma)z(\tau)\| + \|z(\tau)y\| = \|z(\sigma)z(\tau)\| + \|z(\tau)x_1\| \\ \leq& \|z(\sigma)z(\tau)\| + \|z(\tau)z(\sigma)\| + \|z(\sigma)x_1\| \leq C_d + C_d + r(\sigma).\end{aligned}$$

Using the above observation, we can now bound $E_v^{(d)}$ applying the Slivnyak-Mecke formula, the change of variables, and integrating out the constants for the angles. Denoting $\mathcal{B}$ the annulus centered at $z(\sigma)$ and of radii $r(\sigma)$ and $r(\sigma)+2C_d$ (with $\sigma = x_1x_2x_3$), we have:

$$\begin{aligned}E_v^{(d)} &\leq \tfrac{1}{3!}\int_{(\mathbb{R}^2)^3} \mathbb{P}\left(B(\sigma)\cap \mathbf{X}_\gamma = \varnothing\right) \mathbb{P}(\mathcal{B}\cap \mathbf{X}_\gamma \neq \varnothing) \mathbb{1}_{z(\sigma)\in C_\star(v)}\, \mathbb{1}_{r(\sigma)\leq C_r}\, \gamma^3 dx_1\, dx_2\, dx_3 \\ &\leq \frac{2\cdot\gamma^3}{3!}\int_{C_\star(v)}\int_0^{C_r}\int_{\mathbf{S}^3} e^{-\pi\gamma r^2}(1-e^{-4\pi\gamma r C_d})\cdot r^3 \mathcal{A}(u_1,u_2,u_3)\,\mu(du_{1:3})\,dr\,dz \\ &\leq \frac{2\cdot\gamma^3}{3!}\left(\tfrac{10}{8}\right)^2 12\pi^2 \int_0^{C_r} r^3\cdot e^{-\pi\gamma r^2} 4\pi\gamma C_d r dr \\ &= 25\pi^3\gamma^4 C_d \int_0^{C_r} r^4\cdot e^{-\pi\gamma r^2} dr \\ &\leq 25\pi^3\gamma^4 C_d C_r \int_0^{\infty} r^3 e^{-\pi\gamma r^2} dr \\ &= 25\pi^3\gamma^4 C_d C_r \cdot \frac{1}{2\pi^2\gamma^2} \\ &\leq \tfrac{25}{2}\pi\gamma^2 C_d C_r\end{aligned}$$

Finally, we compute bounds for $E_v^{(\alpha)}$. We begin as before, but this time we do not sum over all permutations of the points and instead consider just the angle $\alpha_1((x_1,x_2,x_3),q) = \angle\big(q,x_1,x_2\big)$ for each triple of points $x_1,x_2,x_3 \in (\mathbf{X}_\gamma)^3_{\neq}$. We obtain

$$\begin{aligned}E_v^{(\alpha)} &\leq \int_{(\mathbb{R}^2)^3} \mathbb{P}(B(\sigma)\cap\mathbf{X}_\gamma = \varnothing)\mathbb{1}_{z(\sigma)\in C_\star(v)}\,\mathbb{1}_{r(\sigma)\leq C_r}\,\mathbb{1}_{\alpha_1(\sigma,q)\leq C_\alpha}\,\gamma^3 dx_1\,dx_2\,dx_3 \\ &= 2\gamma^3\int_{C_\star(v)}\int_0^{C_r}\int_{\mathbf{S}^3} e^{-\pi\gamma r^2}\mathbb{1}_{\alpha_1(u_1,u_2,u_3,q)\leq C_\alpha}\cdot r^3\mathcal{A}(u_1,u_2,u_3)\,\mu(du_{1:3})\,dr\,dz \\ &\leq 2\gamma^3\left(\tfrac{10}{8}\right)^2\tfrac{3\sqrt{3}}{4}\int_0^{C_r} r^3 e^{-\pi\gamma r^2}\,dr \quad \int_{\mathbf{S}^3}\mathbb{1}_{\alpha_1(u_1,u_2,u_3,q)\leq C_\alpha}\,\mu(du_{1:3}) \qquad (15) \\ &\leq 2\gamma^3\left(\tfrac{10}{8}\right)^2\tfrac{3\sqrt{3}}{4}\frac{1-e^{-\pi\gamma C_r^2}}{2\pi^2\gamma^2}\int_{\mathbf{S}}\mu(du_1)\int_{\mathbf{S}}4C_\alpha\mu(du_2)\int_{\mathbf{S}}\mu(du_3) \qquad (16) \\ &\leq \tfrac{75\sqrt{3}}{64\pi^2}\gamma\cdot 32\pi^3 C_\alpha = \tfrac{75\sqrt{3}\pi}{2}\gamma C_\alpha\end{aligned}$$

where (15) follows since $\mathcal{A}(u_1,u_2,u_3) \leq \frac{3\sqrt{3}}{4}$ and (16) follows since to have $\angle\big(q,x_1,x_2\big) < t$ we need x_2 to be in the region denoted X_2 in Figure 6 that have measure $4t$.

Finally, we get:

$$\mathbb{P}(E_v) \leq \tfrac{25}{4}\gamma(1+\pi\gamma C_r^2)e^{-\pi\gamma C_r^2} + \tfrac{25}{2}\pi\gamma^2 C_d C_r + \tfrac{75\sqrt{3}\pi}{2}\gamma C_\alpha.$$

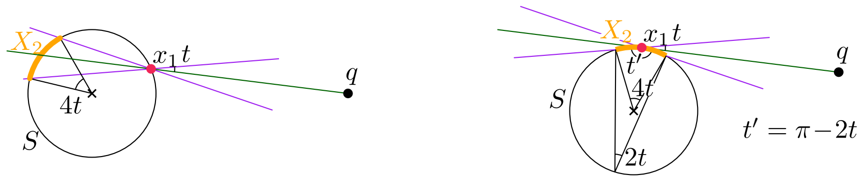


Figure 6: Bounding the measure of small angles.

Using the above, we have that choosing $C_r := \gamma^{-1/3}$, $C_d := \gamma^{-3}$, and $C_\alpha := \gamma^{-3}$ yields $\mathbb{P}(E_v) = O(\gamma^{4/3} e^{-\pi\gamma^{\frac{1}{3}}})$ is asymptotically decreasing as $\gamma \to \infty$. Therefore it suffices to choose γ sufficiently large to guarantee that $\mathbb{P}(E_v) \leq p_2$. In the sequel, γ, C_r, C_d, and C_α are fixed to the present values. $\square$

Proof of Lemma 11, Part (3). When the good event $(E_v)^{\mathrm{c}}$ occurs, (3) is obtained by substituting the above bounds in Equation (11) for any edge with an endpoint in $C_\star(v)$.

$$\begin{aligned}|\Delta_q(\sigma, \cdot)| &\geq 2(\|z(\sigma)q\| - r(\sigma)) \cdot d(\sigma) \cdot \min_{0<i\leq 3} \Big\{ \sin \alpha_i(\sigma, q) \Big\} \\ &\geq 2C_d \cdot C_\alpha \cdot (\|z(\sigma)q\| - C_r)\end{aligned}$$

Notice that, because $C_d \leq \frac{1}{16}$ an edge of $\bar{w}$ that intersects $C(v)$ has an endpoint in $C_\star(v)$. $\square$

8 The Longest Walk

Bounding the number of steps required by the whole walk at once is made complicated since the circle power depends on the distance to the destination. We shall thus split the walk up into a number of *stages*, each of which can be bounded separately. To do this we introduce the following notation, given a walk $w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma))$, we write $\bar{w}[t]$, for $t \in [0, 1]$ to represent the point on the polygonal path $\bar{w}$ at a distance t from $\bar{w}(0)$ normalised so that $\bar{w}[0] = \bar{w}(0)$ and $\bar{w}[1] = \bar{w}(|\bar{w}|) = q$ and taking $\bar{w}[a, b]$ to be a continuous part of $\bar{w}$.

Lemma 12. *Fix $w \in \mathbf{P}^{(q)}(\mathbf{X}_\gamma)$ and $S = \mathbf{A}(\bar{w}[\ell, r])$, for some $0 \leq \ell < r \leq 1$. We shall define the change in circle power over the triangles in the walk w whose centres are in $C(S)$ as,*

$$\Delta_w(S) := \max_{\substack{\sigma \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)), \\ z(\sigma) \in C(S)}} P(\sigma, q) \; - \min_{\substack{\sigma \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)), \\ z(\sigma) \in C(S)}} P(\sigma, q).$$

Then for $L(S, q) := \min\{\, \|xq\| : x \in C(S) \,\} > C_r$,

$$\Big\{ \sum_{v \in S} \mathbb{1}_{E_v^{\mathrm{c}}} \geq t \Big\} \qquad \Longrightarrow \qquad \Delta_w(S) \;\geq\; 2\, C_d \cdot C_\alpha \cdot (L(S, q) - C_r) \cdot \left\lceil \tfrac{t-2}{4} \right\rceil.$$

Proof. Our proof will follow by associating a *delegate* triangle, $\sigma \in w$, with every box $v \in S$ such that E_v^{c} occurs, so that when exiting the delegate triangle, the change in circle power is bounded. Since the circle power is decreasing as the walk advances (by Corollary 10), it will suffice to count the number of unique delegates in order to lower bound the progress made in the circle power. For brevity, we define $z_i := \bar{w}(i)$ for $i = 0, \ldots, |w| - 1$ and $e_i := z_i z_{i+1}$ for $i = 0, \ldots, |w| - 2$. Recall that for $v \in \mathbb{Z}^2$, v may only be *contained* in $S \subseteq \mathbf{A}(\bar{w})$ if one of the line segments, e_i

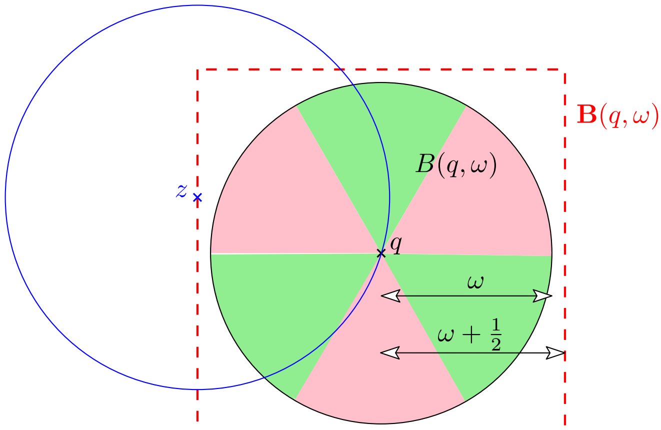


Figure 7: Probability of F_ω.

intersects $C(v)$. Thus, for a given $v \in S$ such that E_v^c occurs, there exists at least one segment, e_x for $x \in 0,\dots,|w|-2$ such that $e_x \cap C(v) \neq \varnothing$. We consider the possible configurations for e_x.

By construction of $C_\star(v)$ and definition of E_v one of the endpoints of e_x, belongs to $C_\star(v)$. Thus, the progress in the circle power made when leaving the triangle dual of this endpoint is at least $2C_d \cdot C_\alpha \cdot (L(S,q) - C_r)$, by Lemma 9. We select the delegate for v to be this vertex of $\bar{w}$.

We have thus found t delegates, assuming that $(E_v)^c$ occurs t times for $v \in S$. However, we note that a single triangle $\sigma \in \mathrm{Del}(\mathbf{X}_\gamma)$ can be a delegate for multiple boxes, $v \in S$, due to overlapping between adjacent boxes. In addition, the very first and very last delegates may not be contained within $C(S)$, though all of the others must be since S is an animal formed from a continuous 'sub-segment' of $\bar{w}$ by assumption. Finally, since one delegate may be shared by at most four boxes, it follows that at least $\lceil (t-2)/4 \rceil$ of the delegates must be unique and the stated result follows. □

We define

$$F_\omega := \Big\{ \forall z \in C(\mathbb{Z}^2 \setminus \mathbf{B}(q,\omega)) B(z, \|zq\|) \cap \mathbf{X}_\gamma \neq \varnothing \Big\}$$

the fact that an empty circle centered ω away from q does not enclose q.

Lemma 13.

$$\mathbb{P}(F_\omega^c) \leq 6\, e^{-\frac{\pi\gamma\omega^2}{6}}.$$

Proof. The probability of F_ω^c can be bounded by noticing that a circle whose center is in $C(\mathbb{Z}^2 \setminus \mathbf{B}(q,\omega)) \subset C(\mathbb{Z}^2 \setminus B(q,\omega))$ and passing through q must enclose one of the six sectors of $B(q,\omega)$, and one of this sector must be empty (see Figure 7). The result follows. □

For positive integers $\ell > r$, we now define

$$S_{\ell,r}(w) := \mathbf{A}\Big(\bar{w}\big[\inf\big\{ t : \bar{w}[t] \in C(\mathbf{B}(q,\ell)) \big\},\ \inf\big\{ t : \bar{w}[t] \in C(\mathbf{B}(q,r)) \big\} \big] \Big).$$

This set represents all of the boxes visited by the walk between the first time the walk intersects $C(\partial\mathbf{B}(q,\ell))$ and the first time it intersects $C(\partial\mathbf{B}(q,r))$. If no such boxes exist we just have

$S_{\ell,r} = \varnothing$. It follows that $S_{\ell,r}(w)$ is an animal on $\mathbf{G}$ since it is a connected subset of the animal $\mathbf{A}(\bar{w})$. In the next lemma we shall bound the number of boxes required to *halve* the distance remaining in the walk. Iteratively applying this bound will be the key in the remaining steps of the proof.

Lemma 14. *For k large enough, and $\omega \leq \frac{k}{2}$*

$$\mathbb{P}\left(\exists w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) : |S_{k,k/2}(w)| \geq \frac{80}{C_d C_\alpha} \cdot k \,\middle|\, F_\omega\right) \leq 8k\, e^{-\frac{40k\sqrt{p_2}}{C_d C_\alpha}}$$

for C_d, C_α, p_2 chosen in Lemma 11.

Proof. We define:

$$H := \bigcup_{u \in \partial\mathbf{B}(q,k)} \left\{ \forall \mathbf{A} \in \mathscr{A}^{(u)}_{\frac{80}{C_d C_\alpha}\cdot k} : \sum_{v \in \mathbf{A}} \mathbb{1}_{E_v} \leq \frac{60k}{C_d C_\alpha} \right\}, \tag{17}$$

the event that any animal of size $\frac{80}{C_d C_\alpha} k$ starting at distance k from q does not have too many bad cells.

The probability of H^c can be bounded using Lemma 8.

$$\mathbb{P}(H^c) \leq 8k\, e^{-\frac{80}{C_d C_\alpha} k \frac{\sqrt{p_2}}{2}}, \tag{18}$$

We now focus on proving the following implication,

$$H \cap F_\omega \quad \Longrightarrow \quad \left\{ \forall w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) : \left|S_{k,k/2}(w)\right| \leq k \cdot \frac{80}{C_d C_\alpha} \right\}, \tag{19}$$

which will follow by contradiction. Suppose that $H \cap F_\omega$ occurs and there exists a path, $w^{\lightning} \in \mathbf{W}(q, \mathrm{Del}(\mathbf{X}_\gamma))$ contradicting the right-hand side of (19). In which case, $|S_{k,k/2}(w^{\lightning})| > \frac{80}{C_d C_\alpha} k$, and as a direct consequence of (17),

$$H \quad \Longrightarrow \quad \left\{ \sum_{v \in S_{k,k/2}(w^{\lightning})} \mathbb{1}_{(E_v)^c} > \frac{20k}{C_d C_\alpha} \right\}. \tag{20}$$

Applying (20) in conjunction with Lemma 12 gives us that the change in circle power is lower bounded,

$$\begin{aligned}
\Delta_{w^{\lightning}}(S_{k,k/2}(w)) \;&\geq\; 2\left(\tfrac{k-1}{2} - C_r\right) \cdot C_d \cdot C_\alpha \cdot \left\lceil \frac{\frac{20k}{C_d C_\alpha} - 2}{4} \right\rceil \\
&\geq\; \frac{\frac{20}{C_d C_\alpha} \cdot C_d \cdot C_\alpha \cdot k^2}{4} - \frac{C_d \cdot C_\alpha(\frac{20}{C_d C_\alpha} \cdot (C_r + \frac{1}{2}) + 1) \cdot k}{2} \\
&\geq\; 5k^2 - O(k) \\
&\geq\; 4k^2,
\end{aligned} \tag{21}$$

for k large enough. We also know that the maximum circle power of q with respect to any triangle whose centre is in $\partial\mathbf{B}(q,k)$ is $(\sqrt{2}k)^2 = 2k^2$ from (9). Since, in addition, F_ω guarantees that no triangle in $w^{\lightning}$ has negative circle power, it follows that

$$\Delta_{w^{\lightning}}(S_{k,k/2}(w)) \leq 2k^2, \tag{22}$$

which is a contradiction with (21). Therefore no such walk $w^{\not\in}$ can exist, and we have proved the implication in (19). To prove the stated result, we condition as follows,

$$\begin{aligned}
&\mathbb{P}\Big(\exists w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) : |S_{k,k/2}(w)| \geq \frac{80}{C_d C_\alpha} \cdot k \,\Big|\, F_\omega \Big) \\
&\leq \mathbb{P}\left(\exists w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) : |S_{k,k/2}(w)| \geq \frac{80}{C_d C_\alpha} \cdot k \,\Big|\, H \cap F_\omega \right) + \mathbb{P}(H^c) \\
&= \mathbb{P}(H^c) && (23) \\
&\leq 8k\, e^{-\frac{40k\sqrt{p_2}}{C_d C_\alpha}} && (24)
\end{aligned}$$

where (23) follows from (19) and (24) follows from (18). □

Lemma 15. *For $\omega \leq \frac{k}{2}$ large enough, we have*

$$\mathbb{P}\left(\exists w \in \mathbf{P}_k^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) : |S_{k,2\omega}(w)| \geq \frac{160}{C_d C_\alpha} \cdot k \,\Big|\, F_\omega \right) \leq e^{-\frac{30\sqrt{p_2}}{C_d C_\alpha}\omega},$$

for C_d, C_α, p_2 chosen in Lemma 11.

Proof. Let $w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma))$ be a walk and $\tau = \lfloor \log_2 \frac{k}{\omega} \rfloor$ we shall consider the following *stages*

$$S_i(w) := S_{k\cdot 2^{-i}, k\cdot 2^{-(i+1)}}(w); \qquad 0 \leq i < \tau.$$

Note that these stages need *not* to be pairwise disjoint, that none of them overlap $C(\mathbf{B}(q,\omega))$, and that $S_{\tau-1}(w)$ ends inside $C(\mathbf{B}(q, 2\omega))$. We observe that

$$\begin{aligned}
&\left\{ \exists w \in \mathbf{P}_k^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) : |S_{k,2\omega}(w)| \geq \frac{160k}{C_d C_\alpha} \right\} \\
&\quad \Longrightarrow \quad \left\{ \exists w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) : \bigcup_{i<\tau} \left\{ |S_i(w)| \geq \frac{160k}{C_d C_\alpha} \cdot \frac{1}{2^{i+1}} \right\} \right\}.
\end{aligned}$$

Which follows since

$$S_{k,2\omega}(w) \subset \bigcup_{0\leq i<\tau} |S_i(w)| \quad \text{and} \quad \frac{160k}{C_d C_\alpha} \sum_{0\leq i<\tau} 2^{-(i+1)} \leq 2\frac{160k}{C_d C_\alpha}.$$

We can now bound the requested probability using Lemma 14.

$$\begin{aligned}
&\mathbb{P}\left(\exists w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)),\ :\ |S_{k,2\omega}(w)| \geq \frac{160k}{C_d C_\alpha} \,\Big|\, F_\omega \right) \\
&\leq \mathbb{P}\left(\exists w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)),\ \exists i<\tau :\ |S_i(w)| \geq \frac{160k}{C_d C_\alpha} \cdot \frac{1}{2^{i+1}} \,\Big|\, F_\omega \right) \\
&\leq \sum_{0\leq i<\tau} \mathbb{P}\left(\exists w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) : |S_{(k\cdot 2^{-i}),(k\cdot 2^{-i})/2}(w)| \geq \frac{80}{C_d C_\alpha} \cdot (k \cdot 2^{-i}) \,\Big|\, F_\omega \right) \\
&\leq \sum_{0\leq i<\tau} 8\cdot 2^{-i} k\, e^{-\frac{40\sqrt{p_2}}{C_d C_\alpha} 2^{-i} k} && (25) \\
&\leq 8\omega \sum_{0\leq i<\infty} 2^i e^{-(\frac{40\sqrt{p_2}}{C_d C_\alpha}\omega) 2^i} \\
&\leq 8\omega \int_1^\infty x e^{-(\frac{40\sqrt{p_2}}{C_d C_\alpha}\omega)x} dx \\
&= \left((\frac{40\sqrt{p_2}}{C_d C_\alpha}\omega) + (\frac{40\sqrt{p_2}}{C_d C_\alpha}\omega)^{-2} \right) e^{-\frac{40\sqrt{p_2}}{C_d C_\alpha}\omega} \leq e^{-\frac{30}{C_d C_\alpha}\omega},
\end{aligned}$$

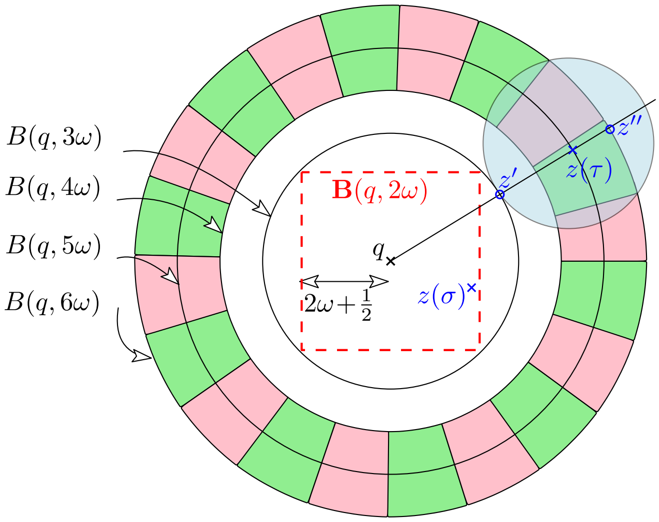


Figure 8: Last step of the walk

for ω large enough. Line (25) follows from Lemma 14. $\square$

Lemma 16. *For ω large enough,*

$$\mathbb{P}\left(\exists w \in \mathbf{P}_k^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) : |S_{2\omega,0}(w)| \geq 121\omega^2\right) \leq 20\, e^{-\pi\gamma\omega^2}$$

Proof. Let σ be a triangle on the walk with center inside $\mathbf{B}(q, 2\omega)$, then $P(\sigma, q) \leq \|z(\sigma)q\|^2 \leq (\sqrt{2}(2\omega + \frac{1}{2}))^2 \leq 9\omega^2$ for ω large enough. For a triangle τ after σ in the walk, we have $P(\tau, q) \leq P(\sigma, q)$ by Corollary 10. Let's assume that $\|z(\tau)q\| \geq 5\omega$ and that $\|z(\tau)q\| - r(\tau) \geq 3\omega$ then $P(\tau, q) = \|qz'\| \cdot \|qz''\| \geq 3 \cdot 5\omega^2 = 15\omega^2$ which is a contradiction (the power is computed considering z', z'' the intersection points of line $qz(\tau)$ and the circle of τ, see Figure 8).

Thus, the circumscribing circle of a triangle τ with $\|z(\tau)q\| \geq 5\omega$ must intersect $B(q, 3\omega)$ and, by consequence, enclose one of the 20 regions subdividing the annulus of radii 4ω and 6ω depicted in Figure 8 must be empty. Such an event can happen only with probability less than $20\, e^{-\frac{\pi\gamma\omega^2(36-16)}{20}} = 20\, e^{-\pi\gamma\omega^2}$.

If no triangle τ with $\|z(\tau)q\| \geq 5\omega$ exists, then the end of the walk can only use triangles with center inside $\mathbf{B}(q, 5\omega$ and $|S_{2\omega,0}(w)| \leq |\mathbf{B}(q, 5\omega| \leq (11\omega)^2$. $\square$

Proof of Proposition 5, Part (1). The proof of the this part now follows easily by combining the results of Lemmas 13, 15, and 16, we get:

$$\begin{aligned}
&\mathbb{P}\left(\exists w \in \mathbf{P}_k^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) : |\mathbf{A}(\bar{w})| \geq \tfrac{160}{C_d C_\alpha} \cdot k + 121\omega^2\right) \\
&\quad \leq \mathbb{P}\left(\exists w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) : S_{k,2\omega}(w) \geq \frac{160}{C_d C_\alpha} k \,\middle|\, F_\omega\right) + \mathbb{P}(F_\omega^{\mathrm{c}}) \\
&\qquad + \mathbb{P}\left(\exists w \in \mathbf{P}^{(q)}(\mathrm{Del}(\mathbf{X}_\gamma)) : S_{2\omega,0}(w) \geq 121\,\omega^2\right) \\
&\quad \leq e^{-\frac{30\sqrt{p_2}}{C_d C_\alpha}\omega} + 6e^{-\frac{\pi\gamma}{6}\omega^2} + 20\, e^{-\pi\gamma\omega^2} \leq e^{-\frac{20\sqrt{p_2}}{C_d C_\alpha}\omega}
\end{aligned}$$

for k large enough, Statement 1 of Proposition 5 follows using $\omega = \sqrt{k}$. $\square$

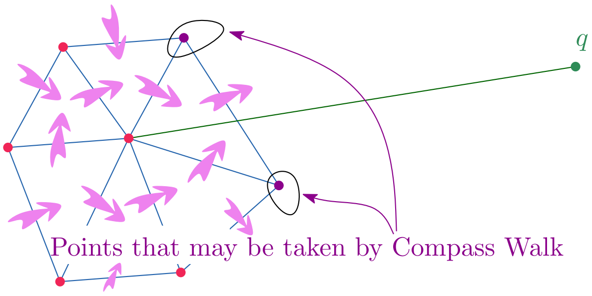


Figure 9: For the proof of Corollary 3

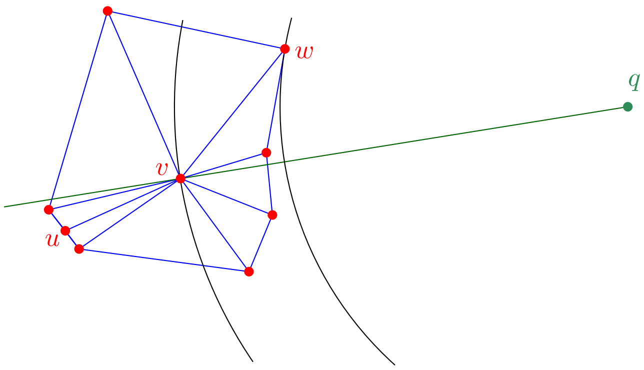


Figure 10: Greedy walk is not a visibility walk

9 Compass and Greedy Walks

Proof of Corollary 3. The result holds because it is possible to associate a unique path in the visibility walk graph with every instance of Compass Walk.

More precisely, in Compass Walk, there are two vertices that may be taken at each step. Note that these are both vertices of a triangle which has two edges oriented 'inwards' and one oriented 'outwards' in the visibility walk graph. It is easy to see that we can always find a path to this triangle once we have entered the *star* of the point in question (see Figure 9). There is therefore always a path in the visibility walk graph associated with every Compass Walk. Since the number of triangles is an upper bound on the number of points, our bound implies a bound on Compass Walk, as stated. □

Unfortunately, there is no easy correspondance between a greedy path and the visibility graph. Let v be a Delaunay vertex, u its predecessor, and w its successor in Greedy Walk towards a query q, as shown in Figure 10, it is possible that non of the triangles incident to vw are reachable from a triangle incident to uv.

So we need to make a direct proof:

Sketch of proof of Corollary 4. The proof is quite similar to the one of Theorem 1. It is actually a bit simpler because the progress is easier to measure by just using the distance from q to the current vertex instead of using the power.

Then it remains to adjust the definition of the bad event E_v to

$$E_v := \bigcup_{x \in \mathbf{X}_\gamma \cap C_v} \left\{ \left\{ d(x) \geq C_d \right\} \cup \left\{ \cos \alpha(x,q) \leq C_\alpha \right\} \right\},$$

where $d(x)$ is the shortest Delaunay edge incident at x and $\alpha(x)$ is the Delaunay edge incident at x with angle closest to $\frac{\pi}{2}$ with xq. $C_d \leq \frac{1}{4}$ is big enough to ensure independence of E_v and E_w when $\|v-w\|_\infty \leq 2$, and any Delaunay edge with an endpoint in $C(v)$ when E_v^c occurs has guaranteed progress. □

10 Concluding remark

In this paper, we prove that the expected complexity of the visibility walk is of order $\sqrt{n}$. Our constants are not very good, nor optimised.

One main question is the extension of this result to bounded domain D considering $\mathrm{Del}(\mathbf{X}_\gamma \cap D)$ instead of $\mathrm{Del}(\mathbf{X}_\gamma)$. Actually the visibility walk in $\mathrm{Del}(\mathbf{X}_\gamma \cap D)$ may have a strange behavior close to the boundary since there are long skinny triangles that allow the walk to “jump” along the boundary. The intuition says that these jumps should shorten the walk, but unfortunately this not always true and it is not possible to prove something on the length of a path in $\mathrm{Del}(\mathbf{X}_\gamma \cap D)$ by finding an equivalent path in $\mathrm{Del}(\mathbf{X}_\gamma$.

The approach in that paper does not generalise directly since the path $\bar{w}$ may go outside D and we have no direct control on its length.

So, we are left with two questions: proving the same result in the bounded case and finding more tractable constant in the complexity.

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